Gauge Fixing and the Faddeev–Popov Procedure

A concise introduction for Yang–Mills theory

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Abstract

The gauge-field path integral is formulated by isolating redundant gauge directions through the Faddeev–Popov procedure. The construction is first illustrated by a finite-dimensional rotational integral and then applied to Yang–Mills theory. The resulting ghost action, covariant-gauge propagator, interaction vertices, and Abelian limit are derived with explicit conventions.

1 Gauge redundancy in the path integral

Let the generators of a compact Lie group satisfy

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c, \qquad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \tag{1.1}$$

and write $A_\mu = A_\mu^a T^a$. We use the mostly-minus metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and

$$D_\mu = \partial_\mu - \mathrm{i} g A_\mu, \qquad F_{\mu\nu} = \frac{\mathrm{i}}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - iig[A_\mu, A_\nu], \tag{1.2}$$

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}. \tag{1.3}$$

For a finite gauge transformation $U(x)$,

$$A_\mu \longmapsto A_\mu^U = U A_\mu U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U) U^{-1}. \tag{1.4}$$

At the free level, after integrating by parts, the quadratic action is

$$S_0[A] = \frac{1}{2}\int \mathrm{d}^4 x\, A^a_\mu (g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu) A^a_\nu. \tag{1.5}$$

In momentum space its kernel is proportional to

$$K^{\mu\nu}(k) = -k^2 g^{\mu\nu} + k^\mu k^\nu, \qquad K^{\mu\nu}(k) k_\nu = 0. \tag{1.6}$$

Thus $K^{\mu\nu}$ has a null direction and cannot be inverted on the full space of fields. This is not a dynamical singularity: configurations related by (1.4) represent the same physical configuration. The naive functional integral integrates over an entire gauge orbit and therefore contains the infinite factor $\mathrm{Vol}(\mathcal{G})$, the volume of the local gauge group [1, 2].

The schematic statement is

$$\int \mathcal{D}A\, e^{\mathrm{i}S[A]} = \mathrm{Vol}(\mathcal{G}) \int_{\text{one representative per orbit}} \mathcal{D}A\, e^{\mathrm{i}S[A]}. \tag{1.7}$$

The Faddeev–Popov construction gives a precise local meaning to this schematic formula [1, 2].

2 The Faddeev–Popov construction

2.1 Finite-dimensional analogue

Consider a rotationally invariant integrand on the plane,

$$W = \int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{r}\, e^{\mathrm{i}S(r)} = \int_0^\infty r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, e^{\mathrm{i}S(r)}. \tag{2.1}$$

Every circle $r =$ constant is an orbit of the rotation group. Choosing the positive x axis, $\theta = 0$, as one representative from each orbit gives

$$W = 2\pi \int_0^\infty r\,\mathrm{d}r\, e^{\mathrm{i}S(r)}. \tag{2.2}$$

The factor 2π is the group volume.

To express the same idea for a more general condition, let $g(r, \theta) = 0$ define a curve that intersects each circular orbit once, and let $\boldsymbol{r}_\phi = (r, \theta + \phi)$. If the zero is simple and unique on $0 \le \phi < 2\pi$, then

$$1 = \Delta_g(\boldsymbol{r}) \int_0^{2\pi} \mathrm{d}\phi\, \delta\big(g(\boldsymbol{r}_\phi)\big), \qquad \Delta_g(\boldsymbol{r}) = \left|\frac{\partial g(\boldsymbol{r}_\phi)}{\partial \phi}\right|_{g(\boldsymbol{r}_\phi)=0}. \tag{2.3}$$

The absolute value is the ordinary Jacobian associated with a delta function. Furthermore,

$$\Delta_g(\boldsymbol{r}_{\phi_0})^{-1} = \int_0^{2\pi} \mathrm{d}\phi\, \delta\big(g(\boldsymbol{r}_{\phi+\phi_0})\big) = \Delta_g(\boldsymbol{r})^{-1}, \tag{2.4}$$

where periodicity permits the shift of the integration variable. Hence the Jacobian is constant along each orbit. Inserting (2.3) into W separates the orbit volume from the integral over the chosen slice. This extends directly to local gauge transformations.

2.2 Functional identity

Choose gauge-fixing functionals $f^a[A](x)$, one for each generator and spacetime point. Ideally, the surface

$$f^a[A](x) = 0 \tag{2.5}$$

intersects each gauge orbit exactly once. Define

$$\mathcal{M}^{ab}(x, y; A) = \left.\frac{\delta f^a[A^\omega](x)}{\delta \omega^b(y)}\right|_{\omega=0}, \qquad \Delta_f[A] = \det \mathcal{M}[A]. \tag{2.6}$$

Here $\omega^a(x)$ parametrises a transformation continuously connected to the identity. Locally in field space,

$$1 = \Delta_f[A] \int \mathcal{D}\omega\, \delta\big[f[A^\omega]\big]. \tag{2.7}$$

This is the Faddeev–Popov identity [1, 2].

In (2.6), the gauge condition is first varied along the gauge orbit. The functional derivative is then evaluated at the point at which that orbit intersects the gauge slice. Since the determinant is constant along an orbit, the representative may be relabelled as A after a change of variables in the path integral.

To see the orbit invariance explicitly, define

$$I[A] = \int \mathcal{D}\omega\, \delta[f(A^{\omega})]. \tag{2.8}$$

For a fixed transformation ω_0, group composition and invariance of the group measure imply

$$I[A^{\omega_0}] = \int \mathcal{D}\omega\, \delta[f((A^{\omega_0})^{\omega})] = \int \mathcal{D}\omega'\, \delta[f(A^{\omega'})] = I[A]. \tag{2.9}$$

Consequently $\Delta_f[A] = I[A]^{-1}$ is constant along the orbit. This is the functional version of the angular shift in the previous section.

Inserting (2.7) into the source-free Yang–Mills integral and changing $A^{\omega} \to A$ gives

$$\mathcal{Z}[0] = \mathrm{Vol}(\mathcal{G}) \int \mathcal{D}A\, \Delta_f[A]\, \delta[f(A)] e^{\mathrm{i}S_{\mathrm{YM}}[A]}. \tag{2.10}$$

The field-independent group volume is absorbed into the normalization. With a source, the standard gauge-fixed generating functional is

$$\mathcal{Z}[J] = \mathcal{N} \int \mathcal{D}A\, \Delta_f[A]\, \delta[f(A)] \exp\left\{\mathrm{i}S_{\mathrm{YM}}[A] + \mathrm{i}\int \mathrm{d}^4x\, J^{a\mu} A^a_\mu\right\}. \tag{2.11}$$

The source term itself need not be gauge invariant; it is a device for generating gauge-field correlation functions in the chosen gauge.

Gribov copies. For non-Abelian gauge theories, a simple local condition such as $\partial^\mu A^a_\mu = 0$ need not intersect every orbit exactly once. The additional intersections are Gribov copies. The perturbative Faddeev–Popov construction is understood locally near the trivial field, where this issue does not obstruct ordinary perturbation theory [3].

3 Covariant gauge and the effective action

3.1 Exponentiation of the determinant

For Grassmann-valued scalar fields c^a and $\bar{c}^a$,

$$\det \mathcal{M}[A] = \mathcal{N}_c \int \mathcal{D}\bar{c}\mathcal{D}c\, \exp\left[\mathrm{i}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \bar{c}^a(x)\mathcal{M}^{ab}(x,y;A)c^b(y)\right]. \tag{3.1}$$

The ghosts are Lorentz scalars but anticommute. They are auxiliary quantum fields rather than physical asymptotic particles; their loop contributions remove unphysical gauge-field degrees of freedom [1, 2].

3.2 Gaussian gauge fixing

Instead of enforcing only $f^a[A] = 0$, insert

$$1 = \Delta_f[A] \int \mathcal{D}\omega \, \delta\big(f^a[A^\omega] - B^a\big) \tag{3.2}$$

and average over B^a with the field-independent Gaussian

$$\int \mathcal{D}B \, \exp\left[-\frac{\mathrm{i}}{2\xi} \int \mathrm{d}^4x \, B^a B^a\right]. \tag{3.3}$$

The result is the gauge-fixing action

$$S_{\mathrm{gf}} = -\frac{1}{2\xi} \int \mathrm{d}^4x \, f^a[A] f^a[A]. \tag{3.4}$$

Thus

$$\mathcal{Z}[J] = \mathcal{N} \int \mathcal{D}A \, \mathcal{D}\bar{c} \, \mathcal{D}c \, \exp\left\{\mathrm{i}S_{\mathrm{eff}} + \mathrm{i} \int \mathrm{d}^4x \, J^{a\mu} A^a_\mu\right\}, \qquad S_{\mathrm{eff}} = S_{\mathrm{YM}} + S_{\mathrm{gf}} + S_{\mathrm{gh}}. \tag{3.5}$$

For the linear covariant gauge, take

$$f^a[A] = \partial^\mu A^a_\mu. \tag{3.6}$$

Writing $U = e^{\mathrm{i}g\omega^a T^a}$, equation (1.4) gives

$$\delta_\omega A^a_\mu = -(D_\mu \omega)^a, \qquad (D_\mu \omega)^a = \partial_\mu \omega^a + g f^{acb} A^c_\mu \omega^b. \tag{3.7}$$

Therefore

$$\delta f^a(x) = -\partial^\mu (D_\mu \omega)^a(x), \tag{3.8}$$

$$\mathcal{M}^{ab}(x, y; A) = -\partial^\mu_x D^{ab}_\mu(x) \, \delta^{(4)}(x - y). \tag{3.9}$$

An overall field-independent sign in the determinant is absorbed into the normalization. The effective Lagrangian is therefore

$$\boxed{\mathcal{L}_{\mathrm{eff}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 - \bar{c}^a \, \partial^\mu (D_\mu c)^a.} \tag{3.10}$$

This is the standard Faddeev–Popov Lagrangian in a linear covariant gauge [1, 2]. Integrating the ghost term by parts gives the equivalent form

$$\mathcal{L}_{\mathrm{gh}} \simeq (\partial^\mu \bar{c}^a)(\partial_\mu c^a) + g f^{acb} (\partial^\mu \bar{c}^a) A^c_\mu c^b, \tag{3.11}$$

where $\simeq$ denotes equality up to a total derivative. The first term is the ghost kinetic term and the second is the ghost–antighost–gluon interaction.

4 Propagators and interaction vertices

4.1 Propagators

The quadratic gauge-field Lagrangian is

$$\mathcal{L}_{A,2} = \frac{1}{2} A^a_\mu \left[g^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu \right] A^a_\nu. \tag{4.1}$$

Using the transverse and longitudinal projectors

$$P_{\mathrm{T}}^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \qquad P_{\mathrm{L}}^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}, \tag{4.2}$$

the inverse kernel is immediate. The Feynman propagator is

$$\boxed{D^{ab}_{\mu\nu}(k) = \frac{-\mathrm{i}\delta^{ab}}{k^2 + \mathrm{i}0}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 + \mathrm{i}0} \right].} \tag{4.3}$$

Feynman gauge is $\xi = 1$ and Landau gauge is the limit $\xi \to 0$. The ghost propagator obtained from (3.11) is

$$D^{ab}_{\mathrm{gh}}(k) = \frac{\mathrm{i}\delta^{ab}}{k^2 + \mathrm{i}0}. \tag{4.4}$$

The propagators follow from the conventional covariant-gauge quadratic action [1, 2].

4.2 Interaction vertices

Expanding the field strength,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{4.5}$$

shows that Yang–Mills theory contains cubic and quartic gauge interactions:

$$\mathcal{L}_{3A} = -g f^{abc} (\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu}, \tag{4.6}$$

$$\mathcal{L}_{4A} = -\frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}, \tag{4.7}$$

$$\mathcal{L}_{\bar{c}cA} = g f^{acb} (\partial^\mu \bar{c}^a) A^c_\mu c^b. \tag{4.8}$$

With all momenta incoming, the three-gluon vertex for (a, μ, p), (b, ν, q), and (c, ρ, r) is

$$g f^{abc} \left[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-r)_\mu + g_{\rho\mu}(r-p)_\nu \right]. \tag{4.9}$$

The four-gluon vertex is

$$\begin{aligned} -\mathrm{i}g^2 \big[& f^{abe} f^{cde} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + f^{ace} f^{bde} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}) \\ & + f^{ade} f^{bce} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) \big]. \end{aligned} \tag{4.10}$$

The ghost–antighost–gluon vertex is proportional to $g f^{abc} p_\mu$, where p is the momentum entering through the antighost line. The placement of an overall i and the sign of the last two rules depend on the Fourier-transform and vertex conventions; the unambiguous starting point is the Lagrangian (3.10).

Ghosts couple only through a non-Abelian structure constant. Their Grassmann character also assigns an additional minus sign to every closed ghost loop [2].

5 Abelian limit

For $U(1)$, all structure constants vanish and

$$A^{\omega}_{\mu} = A_{\mu} - \partial_{\mu}\omega, \qquad \mathcal{M}(x,y) = -\partial^2\delta^{(4)}(x-y). \tag{5.1}$$

The Faddeev–Popov determinant is therefore independent of A_{μ}. It can be absorbed into the overall normalization. Equivalently,

$$\mathcal{L}_{\mathrm{gh}} = -\bar{c}\,\partial^2 c \tag{5.2}$$

describes free ghosts with no interaction vertex. Their vacuum functional is a source-independent constant, so they may be integrated out. This is the precise perturbative sense in which QED is ghost free. The decoupling follows directly from the field independence of the Abelian Faddeev–Popov determinant [1].

The gauge-fixed Abelian Lagrangian is simply

$$\mathcal{L}_{\mathrm{QED,gauge}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^{\mu}A_{\mu})^2, \tag{5.3}$$

and the photon propagator is (4.3) without the colour factor δ^{ab}.

References

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[3] S. Weinberg, *The Quantum Theory of Fields, Vol. II: Modern Applications* (Cambridge University Press, Cambridge, 1996).