NOTES ON

Quantum Field Theory

These notes are an unoriginal collection drawn from standard textbooks and lecture notes that I have gone over while trying to learn quantum field theory.

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Contents

Conventions at a glance

Throughout, Greek indices $\mu, \nu, \ldots$ run over spacetime components $0, 1, 2, 3$. The Latin indices $i, j, \ldots$ run over spatial components $1, 2, 3$, and repeated upper–lower index pairs are summed. The conventions are

$$\hbar = c = 1, \qquad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \qquad x^\mu = (t, \boldsymbol{x}).$$

The metric choice is sometimes called the *mostly-minus* convention.

1 Mathematical tools

This chapter collects the mathematical language used throughout these notes. Units and dimensional conventions are followed by groups, representations, and the generators of continuous transformations. These ideas are then applied to Minkowski spacetime and the Poincaré group. The chapter concludes with the finite-dimensional Lorentz representations carried by covariant fields and the unitary Poincaré representations that describe relativistic one-particle states.

1.1 Units, notation, and dimensional analysis

1.1.1 Natural units

Relativistic quantum theory contains two universal conversion constants: c and $\hbar$. In natural units one sets

$$c = \hbar = 1, \tag{1.1.1}$$

so time, length, mass, momentum, and energy may all be expressed in terms of a single base unit, with conversion factors suppressed. Thus

$$[E] = [p] = [m], \qquad [t] = [x] = [E]^{-1}. \tag{1.1.2}$$

For restoring ordinary units, two useful identities are

$$\hbar c \simeq 0.19732698\ \mathrm{GeV\,fm}, \qquad \hbar \simeq 6.58212 \times 10^{-25}\ \mathrm{GeV\,s}. \tag{1.1.3}$$

Consequently,

$$1\ \mathrm{GeV}^{-1} \simeq 0.1973\ \mathrm{fm} \simeq 6.582 \times 10^{-25}\ \mathrm{s}. \tag{1.1.4}$$

1.1.2 Mass dimension

It is convenient to write $[X] = n$ when the quantity X has mass dimension n, meaning $X \sim (\mathrm{mass})^n$. Since the phase $\mathrm{e}^{\mathrm{i}S}$ must be dimensionless,

$$[S] = 0. \tag{1.1.5}$$

In d spacetime dimensions,

$$[\mathrm{d}^d x] = -d, \qquad [\partial_\mu] = 1, \qquad [\mathcal{L}] = d, \tag{1.1.6}$$

because $S = \int \mathrm{d}^d x\, \mathcal{L}$.

The kinetic terms then determine the canonical dimensions of fields. For a real scalar, a Dirac spinor, and a gauge field (all of which will be introduced subsequently),

$$\mathcal{L}_{\text{scalar}} \supset \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi), \qquad [\phi] = \frac{d-2}{2}, \tag{1.1.7}$$

$$\mathcal{L}_{\text{Dirac}} \supset \bar{\psi}\, \mathrm{i}\gamma^\mu \partial_\mu \psi, \qquad [\psi] = \frac{d-1}{2}, \tag{1.1.8}$$

$$\mathcal{L}_{\text{gauge}} \supset -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad [A_\mu] = \frac{d-2}{2}. \tag{1.1.9}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In four dimensions this becomes

$$[\phi] = 1, \qquad [\psi] = \frac{3}{2}, \qquad [A_\mu] = 1. \tag{1.1.10}$$

If an interaction is written as $g_{\mathcal{O}}\mathcal{O}$, where $[\mathcal{O}] = \Delta_{\mathcal{O}}$, then

$$[g_{\mathcal{O}}] = d - \Delta_{\mathcal{O}}. \tag{1.1.11}$$

This elementary relation anticipates the classification of interactions in effective field theory:

Interaction in $d = 4$	Operator dimension	Coupling dimension
$\frac{\lambda}{4!}\phi^4$	4	$[\lambda] = 0$
$y\,\phi\bar{\psi}\psi$	4	$[y] = 0$
$\frac{g_6}{6!}\phi^6$	6	$[g_6] = -2$
$\frac{c}{\Lambda}\phi F_{\mu\nu}F^{\mu\nu}$	5	$[c/\Lambda] = -1$

Couplings of positive, zero, and negative mass dimension are called relevant, marginal, and irrelevant at the classical level, respectively. Quantum corrections refine this classification; see, for example, Refs. [2, 3].

Example: the dimension of a scalar coupling in general d

Consider $\mathcal{L}_{\text{int}} = -g_n\phi^n/n!$. Since $[\phi] = (d-2)/2$,

$$[g_n] = d - \frac{n(d-2)}{2}. \tag{1.1.12}$$

For $n = 4$, $[g_4] = 4 - d$. Thus ϕ^4 is classically marginal in four dimensions, relevant below four dimensions, and irrelevant above four dimensions.

1.2 Groups and representations

Symmetry transformations are naturally organized into groups. The abstract definition separates the algebraic relations among transformations from the particular objects on which they act.

1.2.1 Definition of a group

A *group* is a set G equipped with a binary operation

$$G \times G \longrightarrow G, \qquad (g_1, g_2) \longmapsto g_1 g_2, \tag{1.2.1}$$

such that

1. **Closure:** $g_1g_2 \in G$ for every $g_1, g_2 \in G$.
2. **Associativity:** $(g_1g_2)g_3 = g_1(g_2g_3)$ for every $g_1, g_2, g_3 \in G$.
3. **Identity:** there is an element $e \in G$ satisfying $eg = ge = g$ for every $g \in G$.
4. **Inverse:** for every $g \in G$ there is an element $g^{-1} \in G$ satisfying $gg^{-1} = g^{-1}g = e$.

A group is *Abelian* if $g_1g_2 = g_2g_1$ for every pair of elements; it is *non-Abelian* otherwise. The order $|G|$ is the number of elements when this number is finite.

The cyclic group $\mathbb{Z}_4$. Under addition modulo four,

$$\mathbb{Z}_4 = \{0, 1, 2, 3\}, \qquad a \circ b = (a+b) \bmod 4. \tag{1.2.2}$$

The identity is 0, the inverse of 1 is 3, and the inverse of 2 is 2. Every element is generated by repeated addition of 1, so $\mathbb{Z}_4$ is a finite cyclic Abelian group.

The group $U(1)$. The complex phases

$$U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}, \tag{1.2.3}$$

form an Abelian group under multiplication. Because $e^{i(\theta+2\pi)} = e^{i\theta}$, its group manifold is a circle. A complex scalar transforms as $\phi \mapsto e^{iq\theta}\phi$. With the 2π periodicity chosen above, the one-dimensional representations have $q \in \mathbb{Z}$; an overall normalization may later be absorbed into the definition of charge.

The matrix groups $U(n)$ and $SO(n)$. Two important continuous groups are

$$U(n) = \{U \in \mathrm{Mat}(n, \mathbb{C}) \mid U^{\dagger}U = \mathbf{1}\}, \tag{1.2.4}$$
$$SO(n) = \{R \in \mathrm{Mat}(n, \mathbb{R}) \mid R^{\mathsf{T}}R = \mathbf{1},\ \det R = 1\}. \tag{1.2.5}$$

Matrix multiplication is the group operation. The first preserves the Hermitian inner product on $\mathbb{C}^n$; the second preserves the Euclidean inner product and orientation on $\mathbb{R}^n$. For $n \geq 2$, $U(n)$ is non-Abelian, and for $n \geq 3$, $SO(n)$ is non-Abelian. The subgroup of $U(n)$ with unit determinant is denoted $SU(n)$.

Groups as transformations

If a group acts on a space of fields or states, composition of transformations must reproduce the group product. This simple requirement is precisely what a representation implements.

1.2.2 Representations

A d-dimensional linear *representation* of a group G is a map

$$D : G \longrightarrow GL(d, \mathbb{F}) \tag{1.2.6}$$

such that

$$D(g_1g_2) = D(g_1)D(g_2), \qquad D(e) = \mathbf{1}_d, \qquad D(g^{-1}) = D(g)^{-1}. \tag{1.2.7}$$

Here $\mathbb{F}$ is usually $\mathbb{R}$ or $\mathbb{C}$. Thus a representation replaces each abstract group element by an invertible linear operator while preserving all multiplication relations.

A representation is *faithful* if $D(g_1) = D(g_2)$ implies $g_1 = g_2$. It is *reducible* if a proper nonzero subspace is invariant under every $D(g)$; otherwise it is *irreducible*. Two representations D and D' are equivalent if a single invertible matrix S exists such that

$$D'(g) = SD(g)S^{-1} \tag{1.2.8}$$

for all $g \in G$. Equivalent representations describe the same group action in different bases. Direct sums combine representations, while tensor products describe the induced action on composite objects:

$$(D_1 \oplus D_2)(g) = D_1(g) \oplus D_2(g), \qquad (D_1 \otimes D_2)(g) = D_1(g) \otimes D_2(g). \tag{1.2.9}$$

Representations of $\mathbb{Z}_3$. Write $\mathbb{Z}_3 = \langle r \mid r^3 = e\rangle$. A representation is fixed once $D(r)$ is chosen, subject to

$$D(r)^3 = \mathbf{1}. \tag{1.2.10}$$

Over $\mathbb{C}$, the three one-dimensional irreducible representations are

$$D_k(r) = \omega^k, \qquad \omega = \mathrm{e}^{2\pi\mathrm{i}/3}, \qquad k = 0, 1, 2. \tag{1.2.11}$$

The case $k = 0$ is the trivial representation.

A faithful two-dimensional real representation is furnished by rotations of the plane through

120°:

$$D(r)=\begin{pmatrix}\cos(2\pi/3) & -\sin(2\pi/3)\\ \sin(2\pi/3) & \cos(2\pi/3)\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}, \qquad D(r)^3=\mathbf{1}_2. \tag{1.2.12}$$

This representation is irreducible over $\mathbb{R}$, since no real line is left invariant by a nontrivial 120° rotation. Over $\mathbb{C}$ it is reducible: in a complex basis it becomes $\mathrm{diag}(\omega,\omega^2)$.

1.3 Lie groups and Lie algebras

A *Lie group* is simultaneously a group and a smooth manifold, with group multiplication and inversion given by smooth maps. Its continuous parameters make it possible to study transformations infinitesimally near the identity. The corresponding infinitesimal generators form the *Lie algebra* of the group.

1.3.1 Generators and finite transformations

In a matrix representation, a group element connected to the identity may be written locally as

$$D(g(\boldsymbol{\alpha}))=\exp(\mathrm{i}\alpha^a T_a)\,, \tag{1.3.1}$$

where the real parameters α^a vanish at the identity and the matrices T_a are the generators in that representation. Infinitesimally,

$$D(g(\boldsymbol{\alpha}))=\mathbf{1}+\mathrm{i}\alpha^a T_a+\mathcal{O}(\alpha^2), \qquad T_a=\frac{1}{\mathrm{i}}\left.\frac{\partial D(g)}{\partial\alpha^a}\right|_{\boldsymbol{\alpha}=0}. \tag{1.3.2}$$

The overall factor of i is conventional and is especially convenient for unitary representations. If $D(g)$ is unitary, then $D(g)^\dagger D(g)=\mathbf{1}$. Expanding this relation to first order gives

$$T_a^\dagger=T_a; \tag{1.3.3}$$

hence the generators of a unitary representation may be chosen Hermitian.

The commutator of two generators is again a generator and defines the structure constants $f_{ab}{}^c$:

$$[T_a,T_b]=\mathrm{i}f_{ab}{}^c T_c. \tag{1.3.4}$$

Antisymmetry of the commutator implies $f_{ab}{}^c=-f_{ba}{}^c$. Associativity of matrix multiplication implies the Jacobi identity,

$$[T_a,[T_b,T_c]]+[T_b,[T_c,T_a]]+[T_c,[T_a,T_b]]=0, \tag{1.3.5}$$

or, equivalently,

$$f_{ab}{}^d f_{dc}{}^e+f_{bc}{}^d f_{da}{}^e+f_{ca}{}^d f_{db}{}^e=0. \tag{1.3.6}$$

This identity ensures that successive infinitesimal transformations close consistently within the same algebra.

1.3.2 $SU(2)$ as a compact non-Abelian Lie group

The group

$$SU(2) = \{U \in \mathrm{Mat}(2,\mathbb{C}),\ U^\dagger U = \mathbf{1}_2,\ \det U = 1\} \tag{1.3.7}$$

is a three-parameter compact Lie group. In its defining representation, a convenient basis of generators is

$$T_i = \frac{\sigma_i}{2}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3.8}$$

They are Hermitian and traceless and obey

$$[T_i, T_j] = \mathrm{i}\epsilon_{ijk} T_k. \tag{1.3.9}$$

A general element connected to the identity is

$$U(\boldsymbol{\theta}) = \exp\left(\frac{\mathrm{i}}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right) = \cos\frac{\theta}{2}\,\mathbf{1}_2 + \mathrm{i}\,\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2}, \qquad \theta = |\boldsymbol{\theta}|\,. \tag{1.3.10}$$

The nonzero commutators make $SU(2)$ non-Abelian. Topologically its group manifold is the three-sphere S^3, which is compact. Moreover, $SU(2)$ is a double cover of $SO(3)$: the elements U and $-U$ correspond to the same spatial rotation.

1.3.3 A noncompact example: translations of the line

Finite translations of the real line form the Lie group $(\mathbb{R}, +)$. If $T(a)$ translates a function by a real displacement a, choose the active convention

$$(T(a)f)(x) = f(x-a). \tag{1.3.11}$$

Then

$$T(a)T(b) = T(a+b), \qquad T(a)^{-1} = T(-a), \tag{1.3.12}$$

and the representation can be written

$$T(a) = \mathrm{e}^{-\mathrm{i}aP}, \qquad P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}. \tag{1.3.13}$$

The generator P is the momentum operator. The group is noncompact because the parameter a ranges over the unbounded line.

1.4 The Lorentz and Poincaré groups

The isometries of Minkowski spacetime consist of Lorentz transformations and spacetime translations. Their generators form the Poincaré algebra, which organizes both covariant fields and relativistic particle states.

1.4.1 The Lorentz group

The Lorentz group is

$$O(1,3) = \{\Lambda \in GL(4,\mathbb{R}),\ \Lambda^{\mathsf{T}}\eta\Lambda = \eta\}, \qquad \eta = \mathrm{diag}(1,-1,-1,-1). \tag{1.4.1}$$

It preserves the Minkowski inner product and hence the interval x^2. Taking determinants of its defining relation gives $(\det\Lambda)^2 = 1$, so $\det\Lambda = \pm 1$. Its four connected components are distinguished by the signs of $\det\Lambda$ and $\Lambda^0{}_0$. The identity component is the *proper orthochronous Lorentz group*

$$SO^+(1,3) = \{\Lambda \in O(1,3)\ \det\Lambda = 1,\ \Lambda^0{}_0 \geq 1\}. \tag{1.4.2}$$

It preserves spatial orientation and the direction of time. Parity and time reversal lie in the other components.

For a transformation near the identity, write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \mathcal{O}(\omega^2). \tag{1.4.3}$$

Substitution into the Lorentz condition gives

$$(\mathbf{1}+\omega)^{\mathsf{T}}\eta(\mathbf{1}+\omega) = \eta + \omega^{\mathsf{T}}\eta + \eta\omega + \mathcal{O}(\omega^2) = \eta, \tag{1.4.4}$$

and therefore

$$\omega_{\mu\nu} \equiv \eta_{\mu\rho}\omega^\rho{}_\nu = -\omega_{\nu\mu}. \tag{1.4.5}$$

An antisymmetric 4×4 matrix has six independent components. The three ω_{ij} generate spatial rotations and the three ω_{0i} generate boosts.

Introduce six antisymmetric generators $M^{\mu\nu} = -M^{\nu\mu}$. In any finite-dimensional representation,

$$D(\Lambda) = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}\right). \tag{1.4.6}$$

The generators in the four-vector representation are

$$\left(M_V^{\rho\sigma}\right)^\mu{}_\nu = \mathrm{i}\left(\eta^{\mu\rho}\delta^\sigma{}_\nu - \eta^{\mu\sigma}\delta^\rho{}_\nu\right). \tag{1.4.7}$$

Expanding equation (1.4.6) with equation (1.4.7) reproduces $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$.

The commutator of two infinitesimal transformations must again be an infinitesimal Lorentz transformation. Multiplying the matrices in equation (1.4.7), or equivalently imposing closure on an arbitrary representation, gives

$$[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\left(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho}\right). \tag{1.4.8}$$

Defining rotation and boost generators by $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ and $K_i = M^{0i}$ yields

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \qquad [J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k. \tag{1.4.9}$$

The relative minus sign in $[K_i, K_j]$ reflects the noncompact boost directions. Finite-dimensional Lorentz representations carried by fields are therefore generally nonunitary. Quantum states instead carry unitary representations of the full Poincaré group.

1.4.2 The Poincaré group

A Poincaré transformation is an affine map

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}, \tag{1.4.10}$$

where $\Lambda \in O(1,3)$ and $a^{\mu} \in \mathbb{R}^{1,3}$. Composing two such transformations gives

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = \big(\Lambda_2\Lambda_1, a_2 + \Lambda_2 a_1\big). \tag{1.4.11}$$

Consequently, the Poincaré group is the semidirect product

$$ISO(1,3) = \mathbb{R}^{1,3} \rtimes O(1,3), \tag{1.4.12}$$

because Lorentz transformations act on the translation subgroup. The identity is $(\mathbf{1}, 0)$ and $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$.

The action of a Poincaré transformation on a multiplet of fields may be written

$$\big[T(\Lambda, a)\Phi\big]_r(x) = D(\Lambda)_r{}^s \Phi_s\Big(\Lambda^{-1}(x-a)\Big), \tag{1.4.13}$$

where $D(\Lambda)$ acts on the field components. The group law in equation (1.4.11) ensures

$$T(g_2)T(g_1) = T(g_2 g_1).$$

The ten infinitesimal generators are the four translations P^{μ} and the six Lorentz generators $M^{\mu\nu}$. Their algebra follows especially transparently from the scalar representation on functions. With

$$P^{\mu} = -\mathrm{i}\partial^{\mu}, \qquad L^{\mu\nu} = x^{\nu}P^{\mu} - x^{\mu}P^{\nu}, \tag{1.4.14}$$

the elementary commutators are

$$[x^{\mu}, x^{\nu}] = 0, \qquad [P^{\mu}, P^{\nu}] = 0, \qquad [x^{\mu}, P^{\nu}] = \mathrm{i}\eta^{\mu\nu}. \tag{1.4.15}$$

For example,

$$\begin{aligned} [L^{\mu\nu}, P^{\rho}] &= [x^{\nu}P^{\mu} - x^{\mu}P^{\nu}, P^{\rho}] && (1.4.16)\\ &= \mathrm{i}\left(\eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu}\right). && (1.4.17) \end{aligned}$$

Expanding $[L^{\mu\nu}, L^{\rho\sigma}]$ with the same three elementary commutators produces the Lorentz algebra equation (1.4.8). With the index order used there, the complete Poincaré algebra is

$$[P^{\mu}, P^{\nu}] = 0, \qquad [M^{\mu\nu}, P^{\rho}] = \mathrm{i}\left(\eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu}\right). \tag{1.4.18}$$

For fields with Lorentz indices,

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \tag{1.4.19}$$

where $L^{\mu\nu}$ acts on the spacetime argument and $S^{\mu\nu}$ acts on the field components. The matrices $S^{\mu\nu}$ obey the Lorentz algebra and commute with x^{μ} and P^{μ}, so the total generators obey the same Poincaré algebra.

Two operators commute with all ten generators and are therefore constant on each irreducible

Poincaré representation:

$$P^2 = P_\mu P^\mu, \qquad W^2 = W_\mu W^\mu, \qquad W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}. \tag{1.4.20}$$

The vector W^μ is the Pauli–Lubanski vector. The eigenvalues of P^2 and W^2 provide the starting point for the particle classification developed at the end of this chapter.

1.5 Minkowski spacetime and index notation

1.5.1 Events, intervals, and the metric

An event in Minkowski spacetime is represented by the contravariant vector

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z). \tag{1.5.1}$$

The Minkowski metric and its inverse are

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.5.2}$$

Indices are lowered and raised with the metric:

$$x_\mu = \eta_{\mu\nu} x^\nu = (t, -x, -y, -z), \qquad x^\mu = \eta^{\mu\nu} x_\nu. \tag{1.5.3}$$

Thus an upstairs spatial component and its downstairs counterpart differ by a minus sign. The invariant interval is

$$x^2 \equiv x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu = t^2 - \boldsymbol{x}^2. \tag{1.5.4}$$

Intervals are timelike, null, or spacelike according as x^2 is positive, zero, or negative respectively.

1.5.2 Contravariant and covariant objects

An element of the Lorentz group acts linearly on contravariant vectors:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.5.5}$$

with metric preservation expressed by equation (1.4.1). In components, that condition reads $\eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$. An object transforming as in equation (1.5.5) is a contravariant vector. A covector transforms with the inverse matrix:

$$a'_\mu = (\Lambda^{-1})^\nu{}_\mu a_\nu = \Lambda_\mu{}^\nu a_\nu. \tag{1.5.6}$$

More generally, every upper index receives one factor of Λ and every lower index one factor of Λ^{-1}.

For a boost of speed v along the x-direction,

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}. \tag{1.5.7}$$

Direct multiplication verifies equation (1.4.1).

1.5.3 Derivatives and index placement

The spacetime derivative is naturally a covector. By the chain rule,

$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu \partial_\nu. \tag{1.5.8}$$

With the mostly-minus metric,

$$\partial_\mu = (\partial_t, \boldsymbol{\nabla}), \qquad \partial^\mu = (\partial_t, -\boldsymbol{\nabla}). \tag{1.5.9}$$

The d'Alembertian is the Lorentz-invariant differential operator

$$\Box \equiv \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2. \tag{1.5.10}$$

1.5.4 Lorentz-invariant contractions

Let a^μ and b^μ be Lorentz vectors. Their transformed inner product is

$$\begin{aligned} a' \cdot b' &= \eta_{\mu\nu} a'^\mu b'^\nu && (1.5.11)\\ &= \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma a^\rho b^\sigma && (1.5.12)\\ &= \eta_{\rho\sigma} a^\rho b^\sigma = a \cdot b, && (1.5.13) \end{aligned}$$

where the Lorentz condition equation (1.4.1) was used on the third line. In matrix notation, this is simply

$$a'^\top \eta b' = a^\top \Lambda^\top \eta \Lambda b = a^\top \eta b. \tag{1.5.14}$$

Important examples of Lorentz scalars and invariant expressions include

$$\begin{aligned} x^2 &= t^2 - \boldsymbol{x}^2, & p^2 &= E^2 - \boldsymbol{p}^2 = m^2, && (1.5.15)\\ p \cdot x &= Et - \boldsymbol{p} \cdot \boldsymbol{x}, & \partial_\mu \phi\, \partial^\mu \phi &= (\partial_t \phi)^2 - (\boldsymbol{\nabla}\phi)^2, && (1.5.16)\\ F_{\mu\nu} F^{\mu\nu} &= 2(\boldsymbol{B}^2 - \boldsymbol{E}^2), & \bar{\psi}\psi &= \text{Lorentz scalar}. && (1.5.17) \end{aligned}$$

For proper Lorentz transformations, $\det \Lambda = +1$, the integration measure $\mathrm{d}^4 x$ is also invariant. Under the full Lorentz group one may more precisely write $\mathrm{d}^4 x' = |\det \Lambda|\, \mathrm{d}^4 x = \mathrm{d}^4 x$.

Not every object with no free index is an ordinary scalar. For example, $\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$ is a *pseudoscalar*: it changes sign under improper Lorentz transformations such as parity. This distinction becomes important when discrete symmetries are discussed.

1.6 Lorentz representations and relativistic particles

The Lorentz algebra can be reorganized into two commuting angular-momentum algebras. This decomposition classifies the finite-dimensional representations carried by relativistic fields. The unitary irreducible representations of the full Poincaré group then classify one-particle states.

1.6.1 The two commuting $SU(2)$ algebras

Define the complex linear combinations

$$A_i = \frac{1}{2}(J_i + \mathrm{i}K_i), \qquad B_i = \frac{1}{2}(J_i - \mathrm{i}K_i). \tag{1.6.1}$$

Using equation (1.4.9), one finds

$$[A_i, A_j] = \frac{1}{4}\left([J_i, J_j] + \mathrm{i}[J_i, K_j] + \mathrm{i}[K_i, J_j] - [K_i, K_j]\right) \tag{1.6.2}$$

$$= \mathrm{i}\epsilon_{ijk}A_k. \tag{1.6.3}$$

The same substitution for B_i, together with the mixed commutator, gives

$$[B_i, B_j] = \mathrm{i}\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0. \tag{1.6.4}$$

Thus the complexified Lorentz algebra separates into two commuting copies of the angular-momentum algebra, conventionally denoted

$$SU(2)_L \times SU(2)_R. \tag{1.6.5}$$

At the group level, the double cover of $SO^+(1,3)$ is $SL(2,\mathbb{C})$. The notation in equation (1.6.5) records the two commuting algebras that label its finite-dimensional complex representations; the Euclidean rotation group in four dimensions has $SU(2)_L \times SU(2)_R$ as its actual double cover.

Since A_i and B_i commute, an irreducible representation is specified by two independent angular momenta,

$$(j_L, j_R), \qquad j_L, j_R = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \tag{1.6.6}$$

with

$$\boldsymbol{A}^2 = j_L(j_L + 1), \qquad \boldsymbol{B}^2 = j_R(j_R + 1). \tag{1.6.7}$$

Its complex dimension is

$$\dim(j_L, j_R) = (2j_L + 1)(2j_R + 1). \tag{1.6.8}$$

Tensor products reduce independently in the two factors. For example,

$$\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right). \tag{1.6.9}$$

Complex conjugation and parity interchange the two labels: $(j_L, j_R) \leftrightarrow (j_R, j_L)$.

1.6.2 Covariant fields and their Lorentz representations

The most frequently encountered finite-dimensional Lorentz representations are summarized below.

Representation	Dim.	Covariant field or tensor structure
$(0,0)$	1	Scalar or pseudoscalar field ϕ
$(\frac{1}{2},0)$	2	Left-handed Weyl spinor ψ_α
$(0,\frac{1}{2})$	2	Right-handed Weyl spinor $\bar{\chi}^{\dot{\alpha}}$
$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	4	Dirac spinor Ψ
$(\frac{1}{2},\frac{1}{2})$	4	Lorentz four-vector V^μ
$(1,0)$	3	Self-dual antisymmetric tensor
$(0,1)$	3	Anti-self-dual antisymmetric tensor
$(1,0)\oplus(0,1)$	6	General antisymmetric tensor, including $F^{\mu\nu}$
$(1,1)$	9	Symmetric traceless rank-two tensor

A scalar is invariant under the connected Lorentz group,

$$\phi'(x') = \phi(x), \tag{1.6.10}$$

whereas a left-handed Weyl spinor and a right-handed Weyl spinor transform in the two inequivalent fundamental spinor representations. A Dirac spinor is their direct sum. A four-vector arises from their tensor product, which explains the representation identity $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$.

Under the rotation subgroup, generated by $\boldsymbol{J} = \boldsymbol{A}+\boldsymbol{B}$, the representation (j_L, j_R) contains ordinary rotation spins

$$j = |j_L - j_R|\,,\ |j_L - j_R| + 1, \ldots, j_L + j_R. \tag{1.6.11}$$

For example, $(\frac{1}{2},\frac{1}{2})$ decomposes under spatial rotations as $0\oplus 1$, corresponding to the temporal component and the three spatial components of a four-vector.

1.6.3 Unitary Poincaré representations and particle states

Let $|p,\sigma\rangle$ be a simultaneous generalized eigenstate of the four translation generators,

$$P^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle. \tag{1.6.12}$$

A Lorentz transformation moves p^μ along an orbit of fixed p^2. Choose a standard momentum k^μ on that orbit. The subgroup that leaves k^μ fixed is the *little group*; its unitary irreducible representations supply the label σ.

For a massive particle, $p^2 = m^2 > 0$ and $p^0 > 0$. The standard momentum may be chosen as

$$k^\mu = (m, \mathbf{0}). \tag{1.6.13}$$

Its little group is $SO(3)$, or $SU(2)$ on the double cover. The irreducible representations are labelled by spin $s = 0, \frac{1}{2}, 1, \ldots$, and the Poincaré Casimirs take the values

$$P^2 = m^2, \qquad W^2 = -m^2 s(s+1). \tag{1.6.14}$$

There are $2s+1$ spin states in the rest frame.

For a massless particle, $p^2 = 0$ and $p^0 > 0$. A convenient standard momentum is

$$k^\mu = (E, 0, 0, E). \tag{1.6.15}$$

The little group is the two-dimensional Euclidean group $ISO(2)$. In the finite-helicity representations relevant to ordinary particles, its translation-like generators act trivially, leaving rotations around the direction of motion. The Pauli–Lubanski vector is then

$$W^\mu = hP^\mu, \qquad P^2 = W^2 = 0, \tag{1.6.16}$$

where h is the helicity. A parity-invariant spectrum contains both h and $-h$.

The labels (j_L, j_R) specify the Lorentz transformation law of a local field, while (m, s) or $(0, h)$ specify the particle states carried by the Hilbert space. Equations of motion, constraints, and gauge symmetry select the physical particle content within a covariant field representation. A Klein–Gordon field in $(0,0)$ creates spin-zero quanta; a massive vector field in $(\frac{1}{2}, \frac{1}{2})$ carries spin one after its constraint is imposed; a massless gauge potential in the same Lorentz representation carries the two helicities $h = \pm 1$.

2 Classical field theory

Relativistic field theory combines the spacetime symmetries developed in the previous chapter with the variational principles of classical mechanics. A fixed-particle description first reveals its limitations at relativistic energies. Fields and their action principle then lead to the field Euler–Lagrange equations and to the conserved currents associated with continuous symmetries.

2.1 Relativistic single-particle quantum mechanics and its limitations

In nonrelativistic quantum mechanics, the Hamiltonian of a free particle is $H = \boldsymbol{p}^2/(2m)$. A tempting relativistic replacement is

$$H = \sqrt{\boldsymbol{p}^2 + m^2}, \tag{2.1.1}$$

keeping only the positive-energy branch. This apparently reasonable theory contains a deep localization problem. The discussion below follows the standard motivation given in Refs. [1, 2, 3].

Suppose a particle is perfectly localized at the origin at $t = 0$. In the position basis, the amplitude to detect it at $\boldsymbol{x}$ after time t is formally

$$K(t, \boldsymbol{x}) = \left\langle \boldsymbol{x} | \mathrm{e}^{-\mathrm{i}Ht} | \boldsymbol{0} \right\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \exp\left[\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} - \mathrm{i}\sqrt{\boldsymbol{p}^2 + m^2}\, t\right]. \tag{2.1.2}$$

Strict one-particle localization would require this amplitude to vanish for $r \equiv |\boldsymbol{x}| > t$. The positive-energy spectrum instead produces a nonzero spacelike tail.

To see the origin of this result cleanly, introduce the positive-frequency two-point function

$$\Delta^+(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\mathrm{e}^{-\mathrm{i}E_{\boldsymbol{p}}t+\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}, \qquad E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2}. \tag{2.1.3}$$

The kernel in equation (2.1.2) satisfies

$$K(t,\boldsymbol{x}) = 2\mathrm{i}\,\partial_t\Delta^+(x). \tag{2.1.4}$$

For a spacelike separation, define

$$\rho \equiv \sqrt{r^2-t^2} > 0. \tag{2.1.5}$$

The momentum integral can be evaluated in terms of a modified Bessel function:

$$\Delta^+(x) = \frac{m}{4\pi^2\rho}K_1(m\rho), \qquad x^2<0. \tag{2.1.6}$$

Since $K_1(z)$ is nonzero for positive z, both $\Delta^+(x)$ and, for generic t, the transition kernel have an exponentially small but nonzero spacelike tail. At $m\rho \gg 1$,

$$\Delta^+(x) \sim \frac{\sqrt{m}}{4\sqrt{2}\,\pi^{3/2}\rho^{3/2}}\mathrm{e}^{-m\rho}. \tag{2.1.7}$$

The characteristic distance is the Compton wavelength m^{-1}. Attempting to localize a particle on shorter scales requires momenta large enough to make particle production energetically possible, so a fixed-particle-number description has left its domain of validity.

2.2 Classical fields and the action principle

2.2.1 From coordinates to fields

In point-particle mechanics, the dynamical variables are a finite collection of generalized coordinates $q_a(t)$. In field theory, the dynamical variables are fields $\phi_a(x)$, one value at every spacetime point. The discrete label a may distinguish species or internal components; it is not a spacetime index unless explicitly stated.

For a local first-derivative theory, the Lagrangian density is

$$\mathcal{L} = \mathcal{L}(\phi_a, \partial_\mu\phi_a; x), \tag{2.2.1}$$

and the action is

$$S[\phi] = \int_\Omega \mathrm{d}^d x\,\mathcal{L}. \tag{2.2.2}$$

The Lagrangian density may change by a total derivative without changing the bulk equations of motion:

$$\mathcal{L} \longrightarrow \mathcal{L} + \partial_\mu K^\mu. \tag{2.2.3}$$

The action then changes only by a boundary term. Boundary terms are not always physically irrelevant—they matter for boundaries, conserved charges, and gravity—but they do not modify the local Euler–Lagrange equations when the variations obey the assumed boundary conditions.

2.2.2 Deriving the field Euler–Lagrange equations

Consider an arbitrary infinitesimal variation

$$\phi_a(x) \longrightarrow \phi_a(x) + \epsilon\, \delta\phi_a(x), \tag{2.2.4}$$

with $\delta\phi_a = 0$ on the boundary $\partial\Omega$. To first order,

$$\begin{aligned}\delta S &= \int_\Omega \mathrm{d}^d x \left[\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\partial_\mu(\delta\phi_a)\right] && (2.2.5)\\ &= \int_\Omega \mathrm{d}^d x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)\right]\delta\phi_a + \int_{\partial\Omega} \mathrm{d}\Sigma_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a. && (2.2.6)\end{aligned}$$

The boundary term vanishes by assumption. Since the remaining $\delta\phi_a(x)$ are arbitrary, stationarity $\delta S = 0$ implies

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right] = 0. \tag{2.2.7}$$

This is the field-theory Euler–Lagrange equation. It is the direct analogue of its point-particle counterpart,

$$\frac{\partial L}{\partial q_a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = 0. \tag{2.2.8}$$

2.2.3 Higher-derivative Lagrangians

A local Lagrangian density may, in principle, depend on derivatives of arbitrarily high order. If

$$\mathcal{L} = \mathcal{L}(\phi_a, \partial_\mu\phi_a, \partial_\mu\partial_\nu\phi_a), \tag{2.2.9}$$

two integrations by parts give

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} + \partial_\mu\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\partial_\nu \phi_a)} = 0. \tag{2.2.10}$$

The appropriate boundary conditions now fix both $\delta\phi_a$ and its first normal derivative. For derivatives through order N, the compact result is

$$\sum_{n=0}^{N} (-1)^n \partial_{\mu_1} \cdots \partial_{\mu_n} \left[\frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1} \cdots \partial_{\mu_n} \phi_a)}\right] = 0, \tag{2.2.11}$$

where the $n = 0$ term means $\partial\mathcal{L}/\partial\phi_a$.

The predominance of first-derivative Lagrangians in elementary field theory has several reasons:

- A nondegenerate Lagrangian with higher time derivatives generally yields higher-order equations of motion, requires extra initial data, and suffers from the Ostrogradsky instability: its Hamiltonian is unbounded below.
- Additional derivatives increase operator dimension, so such terms are often suppressed by powers of a high scale in an effective field theory.
- First-derivative actions already generate second-order equations, which supply the familiar relativistic wave propagation problem.

These statements have important qualifications. Total derivatives do not add dynamics; constrained or degenerate higher-derivative theories can evade the Ostrogradsky conclusion; and higher-derivative operators are routinely used perturbatively in effective field theory.

2.3 Examples of Euler–Lagrange equations

2.3.1 A real scalar field

Consider

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{2.3.1}$$

The two Euler–Lagrange derivatives are

$$\frac{\partial\mathcal{L}}{\partial\phi} = -V'(\phi), \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi. \tag{2.3.2}$$

Therefore,

$$\Box\phi + V'(\phi) = 0. \tag{2.3.3}$$

For $V(\phi) = m^2\phi^2/2$, this reduces to the Klein–Gordon equation

$$(\Box + m^2)\phi = 0. \tag{2.3.4}$$

A plane wave $\phi \sim \mathrm{e}^{-\mathrm{i}p\cdot x}$ satisfies this equation only if

$$p^2 = m^2, \qquad \text{i.e.} \qquad E^2 = \boldsymbol{p}^2 + m^2. \tag{2.3.5}$$

Thus Lorentz invariance of the action directly produces the relativistic dispersion relation.

For the interacting potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{2.3.6}$$

the equation becomes

$$(\Box + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0. \tag{2.3.7}$$

2.3.2 A complex scalar field

Let

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2. \tag{2.3.8}$$

Although ϕ^* is the complex conjugate of ϕ, they are treated as independent variables during variation (equivalently, a complex scalar may be regarded as two real scalar fields). Varying with respect to ϕ^*,

$$\frac{\partial\mathcal{L}}{\partial\phi^*} = -m^2\phi - \lambda(\phi^*\phi)\phi, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \partial^\mu\phi, \tag{2.3.9}$$

and hence

$$(\Box + m^2)\phi + \lambda(\phi^*\phi)\phi = 0. \tag{2.3.10}$$

Varying with respect to ϕ gives the complex-conjugate equation. The Lagrangian is invariant under the global transformation $\phi \to \mathrm{e}^{\mathrm{i}\alpha}\phi$, which will furnish a conserved current and a conserved charge through Noether's theorem.

2.3.3 The electromagnetic field

The gauge potential A_μ defines the antisymmetric field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.3.11}$$

In the presence of an external current j^μ, take

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu. \tag{2.3.12}$$

The Lagrangian depends on A_ν through the source term and on its derivative through $F_{\mu\nu}$:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -j^\nu, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}. \tag{2.3.13}$$

The Euler–Lagrange equation is therefore

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{2.3.14}$$

which contains Gauss's law and the Ampère–Maxwell law. The remaining two Maxwell equations follow identically from the definition of $F_{\mu\nu}$:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \tag{2.3.15}$$

Taking ∂_ν of equation (2.3.14) gives

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0, \tag{2.3.16}$$

because the derivatives are symmetric in μ, ν while $F^{\mu\nu}$ is antisymmetric. Current conservation is thus required for consistency.

The action is unchanged under

$$A_\mu \longrightarrow A_\mu + \partial_\mu \alpha, \tag{2.3.17}$$

provided the source is conserved (up to a boundary term). This gauge redundancy will later be central to quantizing the electromagnetic field.

2.4 Continuous symmetries and Noether's theorem

Noether's theorem establishes the precise relation between continuous symmetries of the action and conservation laws. Consider a collection of fields $\phi_a(x)$ with

$$S[\phi] = \int \mathrm{d}^d x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a; x). \tag{2.4.1}$$

For an infinitesimal continuous transformation with constant parameter ϵ, write

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon \, \Delta x^\mu, \qquad \phi_a(x) \longrightarrow \phi'_a(x') = \phi_a(x) + \epsilon \, \Delta \phi_a(x). \tag{2.4.2}$$

It is useful to distinguish the total field variation $\Delta\phi_a$ from the variation at fixed spacetime point,

$$\delta_0\phi_a \equiv \phi'_a(x) - \phi_a(x) = \Delta\phi_a - \Delta x^\nu \partial_\nu \phi_a. \tag{2.4.3}$$

Define the canonical momentum density

$$\pi^\mu_a \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}. \tag{2.4.4}$$

A direct variation of the Lagrangian gives the identity

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \pi^\mu_a\right] \delta_0\phi_a + \partial_\mu\left(\pi^\mu_a \delta_0\phi_a\right) + \partial_\mu(\mathcal{L}\,\Delta x^\mu)\,. \tag{2.4.5}$$

Suppose the transformation changes the Lagrangian by at most a total derivative,

$$\delta\mathcal{L} = \partial_\mu K^\mu. \tag{2.4.6}$$

Then the Noether current may be written as

$$j^\mu = \pi^\mu_a \delta_0\phi_a + \mathcal{L}\,\Delta x^\mu - K^\mu. \tag{2.4.7}$$

2.4.1 Weak and strong forms

The relation following from equations (2.4.5) and (2.4.6) is

$$\partial_\mu j^\mu = -\left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]\delta_0\phi_a. \tag{2.4.8}$$

This identity holds without imposing the equations of motion and is often referred to as the *strong* form of Noether's theorem. On shell, the Euler–Lagrange equations set the bracket to zero and one obtains

$$\boxed{\partial_\mu j^\mu = 0,} \tag{2.4.9}$$

which is the usual *weak* or on-shell conservation law.

The associated charge is

$$Q = \int \mathrm{d}^{d-1}x\, j^0. \tag{2.4.10}$$

Using the continuity equation,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int \mathrm{d}^{d-1}x\, \boldsymbol{\nabla}\cdot\boldsymbol{j} = -\left(\oint_{S_\infty} \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{j}\right) \to 0. \tag{2.4.11}$$

Hence Q is conserved provided the flux through spatial infinity vanishes.

2.4.2 Internal symmetries

For an internal transformation the coordinates are unchanged, $\Delta x^\mu = 0$. If the Lagrangian is exactly invariant, $K^\mu = 0$, the current reduces to

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a. \tag{2.4.12}$$

Constant shift symmetry. Consider the free massless scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi. \tag{2.4.13}$$

It is invariant under

$$\phi(x) \longrightarrow \phi(x) + \epsilon, \qquad \delta\phi = 1. \tag{2.4.14}$$

The Noether current is therefore

$$j^\mu = \partial^\mu\phi, \tag{2.4.15}$$

with

$$\partial_\mu j^\mu = \Box\phi = 0 \tag{2.4.16}$$

on shell. A mass term or a nonconstant potential breaks this shift symmetry explicitly.

Global phase symmetry. For the complex (charged) scalar theory in equation (2.3.8), the infinitesimal form of

$$\phi \longrightarrow \mathrm{e}^{\mathrm{i}\alpha}\phi, \qquad \phi^* \longrightarrow \mathrm{e}^{-\mathrm{i}\alpha}\phi^* \tag{2.4.17}$$

is

$$\delta\phi = \mathrm{i}\phi, \qquad \delta\phi^* = -\mathrm{i}\phi^*. \tag{2.4.18}$$

Applying equation (2.4.12) gives

$$j^\mu = \mathrm{i}\left(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right). \tag{2.4.19}$$

Equivalently, one may reverse the overall sign by reversing the convention for the infinitesimal parameter. The conserved charge

$$Q = \int \mathrm{d}^3x\, j^0 \tag{2.4.20}$$

is the Noether charge associated with the global $U(1)$ symmetry.

2.4.3 Spacetime translations and the energy momentum tensor

Spacetime transformations act both on the coordinates and on the fields. For translations,

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon^\mu, \tag{2.4.21}$$

with a scalar field transforming according to

$$\phi'(x') = \phi(x). \tag{2.4.22}$$

At fixed x this gives

$$\delta_0\phi = -\epsilon^\nu\partial_\nu\phi. \tag{2.4.23}$$

The corresponding Noether current can be written

$$j^\mu = -\epsilon_\nu T^{\mu\nu}, \tag{2.4.24}$$

where

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathcal{L} \tag{2.4.25}$$

is the canonical energy–momentum tensor. Translation invariance implies

$$\partial_\mu T^\mu{}_\nu = 0. \tag{2.4.26}$$

The four conserved charges are

$$P_\nu = \int d^3x\, T^0{}_\nu. \tag{2.4.27}$$

Thus time-translation invariance gives conservation of energy,

$$H = P^0 = \int d^3x\, T^{00}, \tag{2.4.28}$$

and the three independent spatial translations give conservation of the three components of momentum,

$$P^i = \int d^3x\, T^{0i}, \qquad i = 1, 2, 3. \tag{2.4.29}$$

These constitute the four conserved charges associated with spacetime translations.

3 Canonical quantization of scalar fields

Canonical quantization promotes the phase-space variables of a classical theory to operators. For a field, the discrete canonical coordinates of mechanics become continuously many variables labelled by position. The free real Klein–Gordon field provides the simplest relativistic realization of this construction.

3.1 From Poisson brackets to quantum commutators

For a mechanical system with generalized coordinates $q_i(t)$ and Lagrangian $L(q_i, \dot{q}_i, t)$, the canonical momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{3.1.1}$$

When the velocities can be expressed in terms of (q_i, p_i), the Hamiltonian is the Legendre transform

$$H(q, p, t) = \sum_i p_i \dot{q}_i - L. \tag{3.1.2}$$

For two phase-space functions $F(q, p)$ and $G(q, p)$, the Poisson bracket is

$$\{F, G\}_{\text{PB}} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right). \tag{3.1.3}$$

The fundamental brackets are

$$\{q_i, q_j\}_{\text{PB}} = 0, \qquad \{p_i, p_j\}_{\text{PB}} = 0, \qquad \{q_i, p_j\}_{\text{PB}} = \delta_{ij}. \tag{3.1.4}$$

Hamilton's equations follow directly:

$$\dot{q}_i = \{q_i, H\}_{\text{PB}}, \qquad \dot{p}_i = \{p_i, H\}_{\text{PB}}. \tag{3.1.5}$$

Canonical quantization replaces q_i and p_i by operators and maps the fundamental Poisson brackets to commutators,

$$q_i \longrightarrow \hat{q}_i, \qquad p_i \longrightarrow \hat{p}_i, \qquad \{\ ,\ \}_{\rm PB} \longrightarrow \frac{1}{\rm i}[\ ,\ ]. \tag{3.1.6}$$

Thus

$$[\hat{q}_i, \hat{q}_j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = {\rm i}\delta_{ij}. \tag{3.1.7}$$

In the Heisenberg picture, these commutators reproduce the quantum equations of motion through $\dot{\hat{F}} = {\rm i}[\hat{H}, \hat{F}]$ when $\hat{F}$ has no explicit time dependence.

3.2 Canonical variables of the Klein–Gordon field

Consider the free real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{3.2.1}$$

The field coordinate at fixed time is $\phi(t, \boldsymbol{x})$, and its canonical conjugate is

$$\pi(t, \boldsymbol{x}) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}(t, \boldsymbol{x})} = \dot{\phi}(t, \boldsymbol{x}). \tag{3.2.2}$$

The Hamiltonian density is

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{3.2.3}$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2, \tag{3.2.4}$$

and hence

$$H = \int {\rm d}^3x\, \mathcal{H}. \tag{3.2.5}$$

The field-theory Poisson bracket is the continuum version of equation (3.1.3):

$$\{F, G\}_{\rm PB} = \int {\rm d}^3x \left[\frac{\delta F}{\delta\phi(\boldsymbol{x})}\frac{\delta G}{\delta\pi(\boldsymbol{x})} - \frac{\delta F}{\delta\pi(\boldsymbol{x})}\frac{\delta G}{\delta\phi(\boldsymbol{x})}\right]. \tag{3.2.6}$$

At equal times this gives

$$\{\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})\}_{\rm PB} = 0, \qquad \{\pi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})\}_{\rm PB} = 0, \tag{3.2.7}$$

and

$$\{\phi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})\}_{\rm PB} = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}). \tag{3.2.8}$$

After quantization, ϕ and π are operator-valued distributions. Hats are conventionally suppressed, and the equal-time canonical commutators are

$$[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = 0, \qquad [\pi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = 0, \tag{3.2.9}$$

$$[\phi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = {\rm i}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}). \tag{3.2.10}$$

The Heisenberg equations generated by equation (3.2.5) reproduce

$$\dot{\phi} = \pi, \qquad \dot{\pi} = (\nabla^2 - m^2)\phi, \tag{3.2.11}$$

and therefore the Klein–Gordon equation $(\Box + m^2)\phi = 0$.

3.3 Fourier modes and the operator decomposition

Expand the field in spatial Fourier modes,

$$\phi(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \widetilde{\phi}(t, \boldsymbol{p}) \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{3.3.1}$$

The Klein–Gordon equation becomes an independent harmonic-oscillator equation for every momentum:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + E_{\boldsymbol{p}}^2\right) \widetilde{\phi}(t, \boldsymbol{p}) = 0, \qquad E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}. \tag{3.3.2}$$

Its general positive- and negative-frequency decomposition is

$$\widetilde{\phi}(t, \boldsymbol{p}) = \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left[a(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}E_{\boldsymbol{p}}t} + a^\dagger(-\boldsymbol{p}) \mathrm{e}^{\mathrm{i}E_{\boldsymbol{p}}t} \right]. \tag{3.3.3}$$

The relative adjoint relation follows from the Hermiticity of a real scalar field, $\phi^\dagger = \phi$. Substituting equation (3.3.3) into equation (3.3.1) and changing $\boldsymbol{p} \to -\boldsymbol{p}$ in the negative-frequency term gives the covariant mode expansion

$$\boxed{\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left[a(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}p\cdot x} \right], \qquad p^0 = E_{\boldsymbol{p}}.} \tag{3.3.4}$$

The factor $(2E_{\boldsymbol{p}})^{-1/2}$ is the conventional relativistic normalization. The next stage of the construction determines the algebra of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ from the equal-time canonical commutators and uses these operators to build the scalar-field Fock space.

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