The N^3LO Mixed Contribution to the Pressure of a Hot and Dense Relativistic QED Plasma

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Motivation and scale separation

- Thermal quantum field theory describes relativistic matter in equilibrium through imaginary time and Matsubara modes. In gauge plasmas, hard charged particles generate collective screening encoded by the hard-thermal-loop (HTL) effective theory [2–6].
- The same framework connects the electron–positron plasma of the early Universe with dense, comparatively cold matter relevant to compact stars.
- Weak coupling separates **hard** momenta $K \sim Q_{\mathrm{hard}} \sim \max(\pi T, |\mu|)$ from **soft** photons $K \sim m_{\mathrm{E}} \sim eQ_{\mathrm{hard}}$.
- At next-to-next-to-next-to-leading order ($\mathrm{N^3LO}$), the overlap gives the **mixed** sector: a soft HTL-resummed photon dressed by hard fermion loops. QED has no separate purely soft sector because photons do not self-interact.
- In this work, we obtain the pressure contribution in the mixed sector at N3LO.

Quantity computed. After subtracting the overlap with the hard expansion, we evaluate [1]

$$p_{\mathrm{mix}} = \frac{T}{2} \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \sum_I g_I \frac{\mathcal{N}_I(K)}{K^2[K^2 + \Pi_I^{\mathrm{LO}}(K)]},$$

$$\mathcal{N}_I(K) = \Pi_I^{\mathrm{LO}}(K)[\Pi_I^{\mathrm{NLO}}(K) - \Pi_I^{\mathrm{Pow}}(K)\Pi_I^{\mathrm{LO}}(K)].$$

Here $I \in \{\mathrm{T}, \mathrm{L}\}$ labels transverse and longitudinal photons, while LO, NLO, and Pow denote the HTL, two-loop, and power-correction pieces of the self-energies. The metric is Euclidean, since we are in imaginary time with $K = (2\pi n T, \mathbf{k})$.

Static and nonstatic Matsubara modes

Static mode. The zero mode is analytic and begins at $\mathcal{O}(e^5)$:

$$p_{\mathrm{mix}}^{(0)} = -\frac{e^2 T m_{\mathrm{E}}^3}{\pi^3} \left[\frac{3}{64} + \frac{N_f}{96}(\mathcal{I}_T - 3)\right].$$

$\mathcal{I}_T$ is a function of T, μ, and the renormalisation scale $\bar{\Lambda}$.

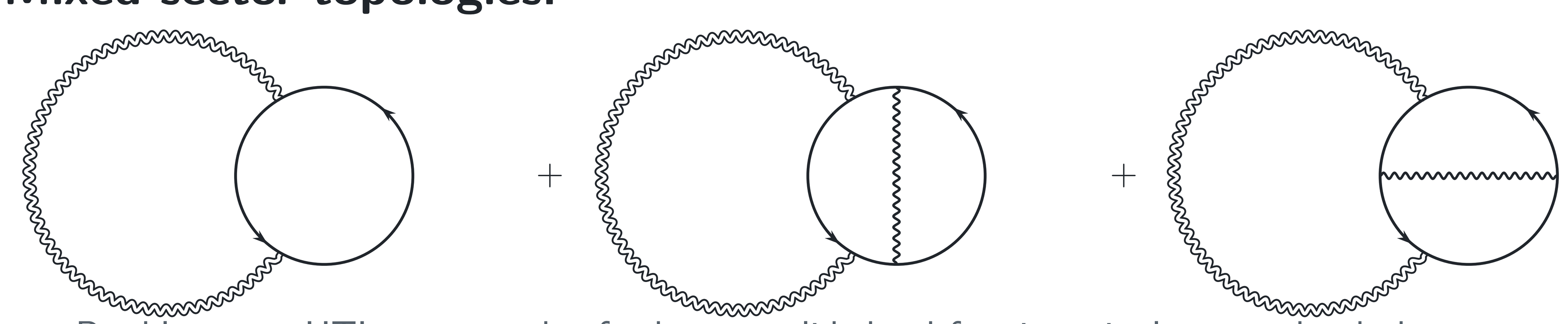
Nonstatic modes. For $n \neq 0$, the Matsubara sum is performed after an angular change of variables $(\tan\phi = \frac{|\omega_n|}{k})$, reduces the pressure to

$$p_{\mathrm{mix}}^{(n\neq 0)} = \int_0^{\pi/2} \mathrm{d}\phi \sum_I w_I(\phi) \mathcal{F}_d(\alpha_I(\phi)),$$

$$\mathcal{F}_d(\alpha) = \frac{1}{2\epsilon} - \mathrm{Re}\,\psi(1 + \mathrm{i}\alpha) + \mathcal{O}(\epsilon).$$

Here $\alpha_I^2(\phi) = \Pi_I^{\mathrm{LO}}(\phi)\cos^2\phi/(2\pi T)^2$, while w_I contains the remaining loop and angular functions.

Mixed-sector topologies.



Double wavy: HTL-resummed soft photon; solid: hard fermion; single wavy: hard photon.

Pole structure. Dimensional regularization separates the mixed pressure as

$$p_{\mathrm{mix}} = \frac{A_{-1}}{\epsilon} + A_0 + \mathcal{O}(\epsilon).$$

The pole is analytic and cancels against the infrared pole of the hard sector; only the one-dimensional digamma integral in A_0 remains numerical.

Limits, interpretation, and outlook

Thermal and cold limits.

- For $\alpha_I \to 0$, the high-temperature expansion is analytic in e^2.
- For $\alpha_I \to \infty$, the cold limit generates an additional $e^6 \ln e$ contribution.

The finite part of $\mathcal{F}_d$ interpolates between

$$\mathcal{F}(\alpha) = \begin{cases} \gamma_E - \zeta(3)\alpha^2 + \mathcal{O}(\alpha^4), & \alpha \ll 1, \\ -\ln\alpha - \dfrac{1}{12\alpha^2} + \mathcal{O}(\alpha^{-4}), & \alpha \gg 1. \end{cases}$$

The large-α expansion reproduces the known zero-temperature result [7].

Future directions.

- Evaluate the remaining finite hard contribution and assemble the complete $\mathrm{N^3LO}$ equation of state at arbitrary T/μ, including its thermodynamic derivatives and limiting regimes.
- Extend the finite-T, finite-μ framework to QCD, where non-Abelian soft contributions introduce additional scales and resummations relevant to hot and dense strongly interacting matter.

References

[1] T. Gorda and S. Majumder, *The N^3LO Mixed Contribution to the Pressure of Hot and Dense Quantum Electrodynamics*, manuscript in preparation (2026).

[2] C. Manuel, J. Soto, and S. Stetina, *The on-shell effective field theory: a systematic tool to compute power corrections to the hard thermal loops*, Phys. Rev. D **94**, 025017 (2016); **96**, 129901(E) (2017), arXiv:1603.05514.

[3] S. Carignano, C. Manuel, and J. Soto, *Power corrections to the HTL effective Lagrangian of QED*, Phys. Lett. B **780**, 308–312 (2018), arXiv:1712.07949.

[4] S. Carignano, M. E. Carrington, and J. Soto, *The HTL Lagrangian at NLO: the photon case*, Phys. Lett. B **801**, 135193 (2020), arXiv:1909.10545.

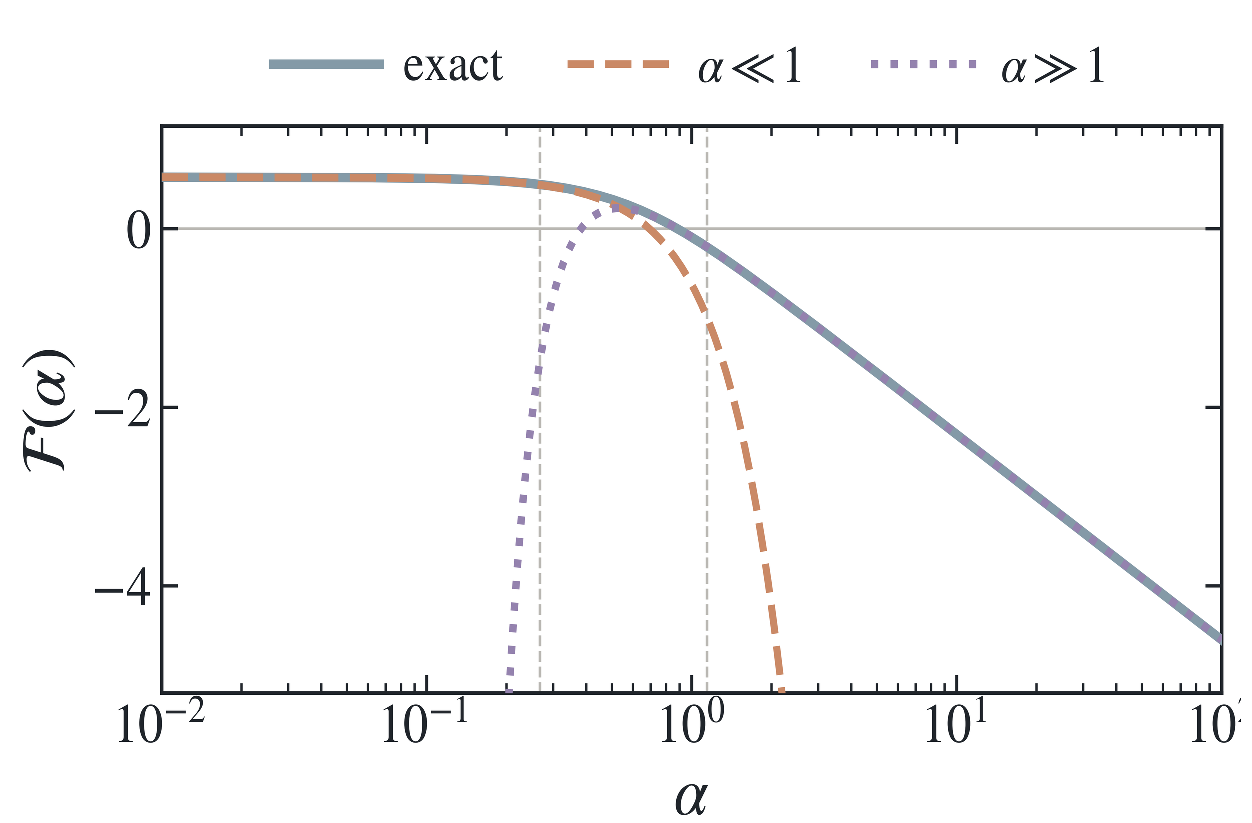
[5] T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Soft photon propagation in a hot and dense medium to next-to-leading order*, Phys. Rev. D **107**, 036012 (2023), arXiv:2204.11279.

[6] T. Gorda, A. Kurkela, R. Paatelainen, S. Säppi, and A. Vuorinen, *Cold quark matter at N^3LO: soft contributions*, Phys. Rev. D **104**, 074015 (2021), arXiv:2103.07427.

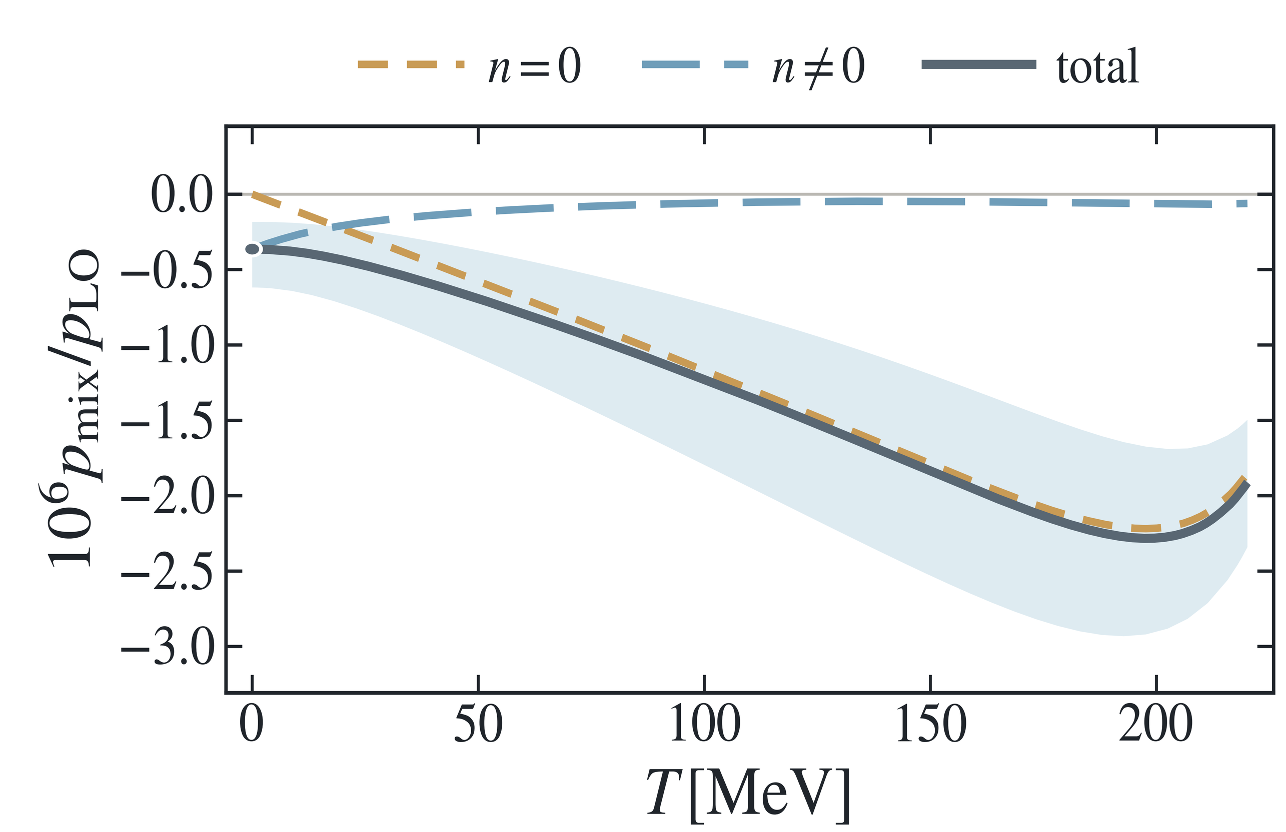
[7] T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Degenerate fermionic matter at N^3LO: Quantum Electrodynamics*, Phys. Rev. D **107**, L031501 (2023), arXiv:2204.11893.

Hot-to-cold crossover

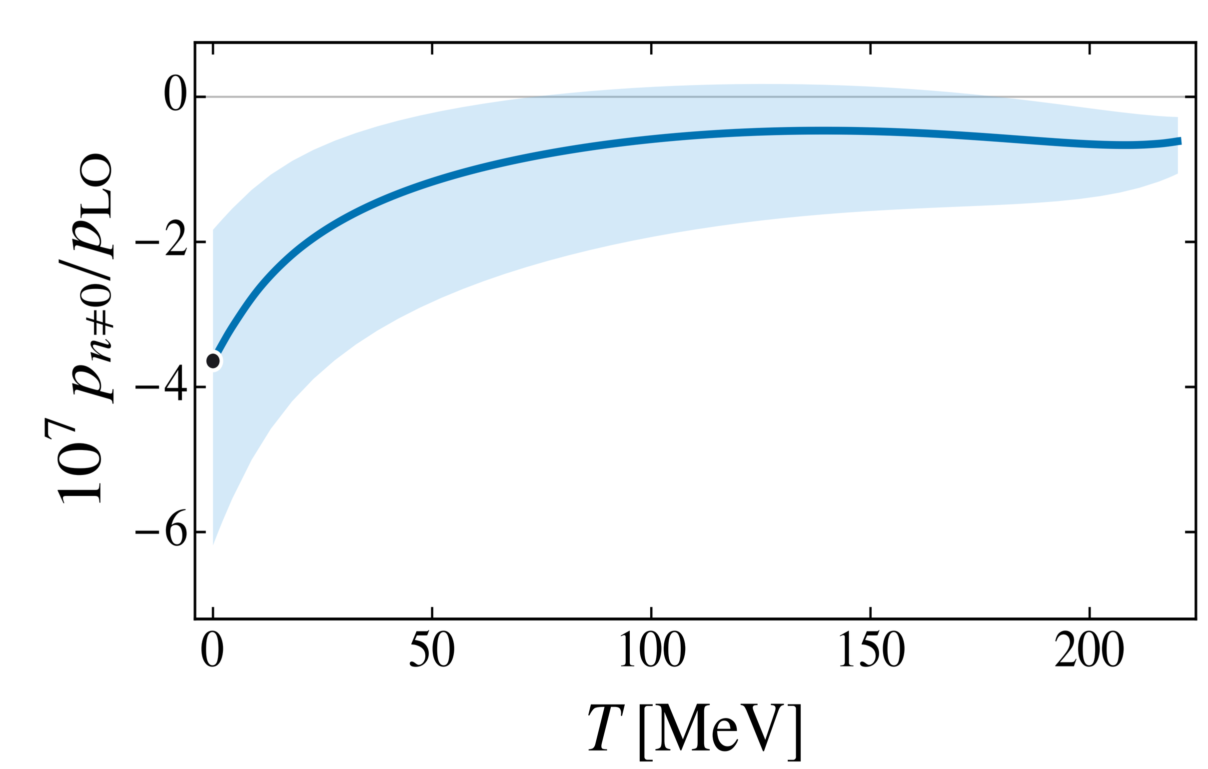
For $N_f = 1$, all pressure panels follow $\sqrt{(2\mu)^2 + (0.723\,4\pi T)^2} = 2$ GeV. The $X \in [1/2, 2]$ band varies $\bar{\Lambda} = \Lambda_h = X(2\ \mathrm{GeV})$.



Finite Matsubara function's interpolation



Total mixed pressure versus T.



Nonzero-mode pressure versus T.